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Griffiths singularities as a consequence of Lifshitz band tails

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Abstract. We consider an $O(n)$-invariant continuum model of a ferromagnet in d dimensions with quenched temperature-like disorder and determine the domain of the Griffiths phase in terms of the parameters of this model. Using the Ornstein–Zernike approximation with respect to thermal fluctuations we investigate the temperature region above the upper boundary of this domain ($T \geq T_{\mathrm{G}}$) for a bounded distribution of disorder, and work out the connection with the corresponding Lifshitz problem. Using different assumptions for the disorder we give an expression for the Griffiths singular part of the free energy and show that an essential singularity develops in the limit $T \searrow T_{\mathrm{G}}$ in zero magnetic field.

1. Introduction

Although known for many years, the status of Griffiths singularities is still unsatisfactory [1]. There are two main reasons why neither theoretical work has given testable predictions, apart from a few exceptions, nor any experiment has been done on this subject so far.

The first reason concerns the poor observability of these singularities. While Griffiths in his original work only proved the existence of singularities in a temperature interval $T_{\mathrm{c}} < T \leq T_{\mathrm{G}}$ above the critical temperature of a random ferromagnet it later became clear that the non-analyticities were merely essential singularities. Thus as far as static quantities are concerned there is no point in looking for the singularities experimentally [2]. An investigation of the dynamic effects, on the other hand, offers a more promising route for experimentalists. The reason is that Griffiths singularities manifest themselves by the appearance of a non-exponential decay of autocorrelation functions, which is slower than the usual exponential decay in regular paramagnetic systems [3–5]. However, suitable experiments have not yet been done.

The second reason concerns the theoretical efforts. In Griffiths singularities, which are typically non-perturbative phenomena, powerful tools similar, for example, to the renormalization group approach to critical phenomena have not yet been developed. There are indeed numerous papers on the subject: papers using heuristic arguments [4, 5] or giving exact solutions for the one-dimensional Ising model [6] and other special models [7, 8], applying instanton techniques [9, 10] or making use of rigorous methods [11]. However, few, if any, give explicit results for the form of the Griffiths singularity.

In an early paper Harris [2] pointed out the qualitative similarity of the Griffiths phase to the occurrence of Lifshitz band tails in disordered electronic systems both

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of which are caused by the existence of arbitrarily large, though rare, regions without disorder. Very recently Nieuwenhuizen [12] considered the two-dimensional Ising model with an anisotropic form for the coupling disorder and he succeeded in finding an explicit relationship between the free energy in a zero magnetic field and the density of states, in particular the Lifshitz tail, of the corresponding electronic system.

In this paper we elaborate on this connection for a general class of continuum models, namely models of the Ginzburg–Landau type with an n-component order parameter field and an $\mathrm{O}(n)$-invariant Hamiltonian, which are well known from the investigation of critical phenomena. Such models with quenched temperature-like disorder are well established for the description of critical phenomena in disordered systems where the disorder variable is usually assumed to have a Gaussian distribution—deviations being irrelevant perturbations in the renormalization group sense (for a review of critical behaviour in disordered systems and an extensive list of references see [13]; and [14] and [15]). In section 2 we will examine this and other assumptions on the disorder to distinguish between important and less important properties characterizing the disorder. Furthermore we restrict ourselves to Gaussian thermal fluctuations and to temperatures T above the Griffiths temperature. The latter restriction aims to extract the Griffiths singularity of the free energy at T_G in zero magnetic field *as a function of* $T - T_G$. The former restriction, the Ornstein–Zernike approximation, allows us to derive an explicit relationship between the free energy and the density of states of the corresponding Lifshitz problem (section 3). Section 4 serves to recall some facts about Lifshitz behaviour and gives the typical form for Lifshitz band tails. Exploiting this we show in section 5 that the free energy has an essential singularity in the limit $T \searrow T_G$. The final section contains a brief summary of our findings and some concluding remarks.

2. Model

We consider a model defined by the reduced Hamiltonian

$$\mathcal{H} = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\tau(x)\phi^2 + \frac{u}{4!}(\phi^2)^2 \right\} \tag{2.1}$$

where $u > 0$, while $\phi = (\phi_1, \ldots, \phi_n)$ is an n-component order parameter field, and the range of integration extends over the whole $\mathbb{R}^d$. The function $\tau(x)$ is a stochastic field representing a kind of locally fluctuating critical temperature. This can be easily seen in the mean field approximation by recalling the familiar relationship $\tau \propto T - T_c$ for pure systems near criticality which may be transferred to the present case as

$$\tau(x) \propto T - T_c(x) \tag{2.2}$$

To determine the properties of $\tau(x)$ it is helpful to imagine our model (2.1) as a model of dilute ferromagnet. According to Griffiths's original considerations [1] the upper temperature bound for the appearance of singularities, the so-called Griffiths temperature T_G, is given by the critical temperature of the pure system. To be more precise, consider a lattice model of a ferromagnet with random nearest-neighbour couplings J_{ij} which are positive and independent, identically distributed according to a distribution $\mathcal{P}(J_{ij})$. The pure system is then, by definition, that one for which all

J_{ij} take on their maximal value allowed by $\mathcal{P}$. In other words: T_G is the maximal critical temperature compatible with $\mathcal{P}$†. This implies that for T_G to be finite the supremum of the support of $\mathcal{P}$ (sup supp $\mathcal{P}$), the maximal value of J_{ij}, has to be finite [5]. From (2.2) we conclude that, for our continuum model (2.1), a finite T_G requires *a distribution for $\tau(x)$ bounded from below.* (Note that a Gaussian distribution is thus excluded!) Therefore, without loss of generality, we may write

$$\tau(x) = \tau + \delta\tau(x) \qquad \delta\tau(x) \geq 0. \tag{2.3}$$

In addition we also assume *the support of $\delta\tau$ to be bounded from above.* On the other hand we are interested in systems which are still translationally invariant—in the mean. So *$\delta\tau(x)$ has to be identically distributed for all $x \in \mathbb{R}^d$.* For the present we will not specify the properties of the stochastic field $\delta\tau$ in more detail and come back to it later.

From this it is clear that the Griffiths temperature T_G corresponds to $\tau_G = 0 + O(u)$. The actual critical temperature of the infinite system, τ_c, consequently is located at a negative value of $\tau : \tau_c < 0$†. Note that the existence of an upper bound for $\delta\tau(x)$ implies the existence of a critical temperature ($\tau_c > -\sup \operatorname{supp} \delta\tau$). Thus our model cannot describe phenomena below the percolation threshold of the originally diluted ferromagnet. But since we are going to investigate the behaviour near T_G this should not matter. However, the interval $\tau_c < \tau \leq \tau_G$ is usually called the 'Griffiths phase'. In the space of temperature τ and magnetic field H the set $\{\tau_c < \tau \leq \tau_G; H = 0\}$ is the line of Griffiths singularities. Especially in the limit $H = 0$, $\tau \searrow \tau_G$, we expect the thermodynamic quantities to exhibit essential singular behaviour, as will be proved later‡

For the following we will confine ourselves to the Ornstein–Zernike approximation ('mean field theory' with respect to the thermal fluctuations). Hence we set $u = 0$ for the future and obtain, as a consequence, $\tau_G = 0$. In addition we will concentrate on the case $\tau \geq 0$ and obtain an essential singularity in the free energy in the previously mentioned limit, $\tau \searrow 0$ (and $H = 0$). The Hamiltonian of our (Gaussian) model can now be rewritten as

$$\begin{aligned} \mathcal{H}_0\{\phi\} &= \int \mathrm{d}^d x \left\{ \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}[\tau + \delta\tau(x)]\phi^2 \right\} \\ &= \sum_{\alpha=1}^{n} \tfrac{1}{2}(\phi_\alpha, G_0^{-1}\phi_\alpha) \end{aligned} \tag{2.4}$$

with $(\varphi, \psi) := \int \mathrm{d}^d x \varphi(x)\psi(x)$ and the inverse propagator

$$G_0^{-1} := -\Delta + \tau + \delta\tau(x) \tag{2.5}$$

† Note that, nevertheless, an infinite disordered system has, if at all, a unique critical temperature T_c depending on $\mathcal{P}$, but 'almost surely' independent of the realization. This is due to the fact that the free energy is a self-averaging quantity (see also [16, 17]).

‡ Other examples of Griffiths singularities in the temperature field are to be found in [7] and [12].

3. Free energy and Lifshitz tails

The reduced free energy per unit volume is

$$\mathcal{F}_0 = -\frac{1}{V} \ln \mathrm{Tr}_{\phi} \exp[-\mathcal{H}_0\{\phi\}]. \tag{3.1}$$

Neglecting a trivial additional constant we immediately arrive at

$$\begin{aligned} \mathcal{F}_0 &= \frac{n}{2V} \ln \det G_0^{-1} \\ &= \frac{n}{2V} \mathrm{Tr} \ln G_0^{-1}. \end{aligned} \tag{3.2}$$

Here, in mean field theory and above T_{G} the n degrees of freedom of the vector spins are completely decoupled and show up in a factor n in front of the free energy. In the thermodynamic limit the trace can be replaced by an integral with measure $V\rho_0(\mu)\mathrm{d}\mu$ where $\rho_0(\mu)$ is the density of states per unit volume of the operator G_0^{-1} given in (2.5)†. For the non-trivial part of the free energy density we thus get

$$\mathcal{F}_0 = \frac{1}{2} n \int_0^{\Lambda} \rho_0(\mu) \ln \mu \, \mathrm{d}\mu. \tag{3.3}$$

Here we have introduced a cutoff Λ to regularize the theory‡. From (3.3) we see that the Griffiths problem can be traced back to the calculation of the density of states $\rho_0(\mu)$ of G_0^{-1}. But G_0^{-1} may be interpreted as a Hamiltonian operator of a disordered electronic system (in the one-particle approximation) characterized by the stochastic potential $V(x) = \tau + \delta\tau(x)$. Thus (3.3) shows that *the Griffiths problem in the mean field approximation at T_{G} is equivalent to the Lifshitz problem for the Hamiltonian operator (2.5)*, analogously to the corresponding results on the diluted two-dimensional Ising model [10, 12].

4. General form of Lifshitz tails

Since Lifshitz's original paper in 1964 [18] a lot of work has been done to determine the density of states for different models and diverse kinds of stochastic potentials (see e.g. [19, 20] and references therein). From these numerous papers we will choose a few examples to crystallize the form of the essential singularity appearing in $\mathcal{F}_0$ for $T \searrow T_{\mathrm{G}}$. From a mathematical point of view it is (because of questions of existence; for a review of the mathematics see [21]) more convenient not to consider the density of states $\rho_0(\mu)$ but the integrated density of states defined by

$$Z_0(\mu) = \lim_{L\to\infty} \frac{1}{L^d} \#\{\text{eigenstates of } G_{0,L,\mathrm{bc}}^{-1} \text{ with eigenvalue } \hat{\mu} \leq \mu\} \tag{4.1}$$

† Expressed in terms of the imaginary part of the Green function the density of states reads: $\rho_0(\mu) = \pi^{-1}\mathrm{Im}\mathrm{Tr}[G_0^{-1} - \mu - \mathrm{i}0]^{-1}$. For another definition see section 4.

‡ As usual in the field-theoretical treatment of statistical mechanics the necessity of an ultraviolet cutoff to obtain well defined physical quantities reflects the existence of a microscopic length-scale a. Therefore one may set $\Lambda = a^{-2}$.

where $G_{0,L,\mathrm{bc}}^{-1}$ is the Hamiltonian (2.5) restricted to a cube of size L^d with some boundary conditions 'bc'. In many cases it is possible to prove rigorously the existence of $Z_0(\mu)$ and its independence of the boundary conditions [21]. The relationship between $Z_0(\mu)$ and $\rho_0(\mu)$ (if ρ_0 exists) is given by $\rho_0(\mu) = \mathrm{d}Z_0(\mu)/\mathrm{d}(\mu) = Z_0'(\mu)$.

Different assumptions abut the stochastic potential $\delta\tau(x)$ may lead (or may not lead) to different expressions for the Lifshitz tail, i.e. the essential singularity appearing in the (integrated) density of states $Z_0(\mu)$ at the lower edge of the spectrum, τ. But the leading asymptotic behaviour of $Z_0(\mu)$ for $\mu \searrow \tau$ can be written in the general form

$$Z_0(\mu) \sim \begin{cases} \exp\{-\kappa|\ln(\mu-\tau)|^s(\mu-\tau)^{-\zeta}\} & \text{for } \mu \searrow \tau \\ 0 & \text{for } \mu \le \tau \end{cases} \tag{4.2}$$

with suitable constants $\kappa, \zeta > 0$ and $s \ge 0$. This will be illustrated later. Prefactors depending algebraically on $(\mu - \tau)$ may be understood as additive logarithmic corrections in the exponent and are therefore omitted.

Surveying the literature we find a variety of investigated models leading to different values for ζ and s. Our aim in the remainder of the section is to give a classification of the models with regard to these exponents. To do this, we have to distinguish between unimportant details of the models, which do not affect the asymptotic behaviour, and important properties, determining ζ and s. Accordingly we distinguish 'technical' aspects from 'physical' ones. To be more precise we introduce two classes of models, namely:

(A) models with a stochastic field $\delta\tau$ which is induced by a *countable set of 'primary' stochastic variables* and can be written as

$$\delta\tau(x) = \sum_{i\in\mathbb{Z}^d} q_i f(x-i) \tag{4.3}$$

where the $q_i \ge 0$ are stochastic variables to be specified later and $f(x) \ge 0$, $x \in \mathbb{R}^d$;

(B) models not reducible to a countable set of stochastic variables. Thus they are characterized by the stochastic properties of $\delta\tau(x)$ itself — as the primary stochastic variables.

This distinction between (A) and (B) is what we call different 'technical' aspects and there are some indications [18–21] that the asymptotic singular behaviour in the limit $\mu \searrow \tau$ does not depend on these differences. Concerning the 'physical' aspects there are, at least, two factors which affect the Lifshitz tails, i.e. the exponents ζ and s in (4.2):

(I) *The range of correlations of disorder* which may be characterized by the cumulant

$$C(x-x') = [\delta\tau(x)\delta\tau(x')]_{\mathrm{av}} - [\delta\tau(x)]_{\mathrm{av}}^2$$

where $[\cdot]_{\mathrm{av}}$ is the expectation value with respect to the distribution of $\delta\tau$. Note that $[\delta\tau(x)]_{\mathrm{av}}$ is a position-independent quantity due to the translational invariance of the distribution of $\delta\tau$ (see above). Now we distinguish between

(α) *short-range correlations*: $C(x) \le c_1|x|^{-\alpha}$ for $\alpha > d+2, c_1 > 0$ and

(β) *long-range correlations*: $C(x) \sim |x|^{-\alpha}$ for $|x| \to \infty, d < \alpha \le d+2$.

Another familiar kind of disorder is given by the 'white noise' case i.e. $C(x) \propto \delta(x)$. From work on Gaussian distributed disorder one finds that this delta-correlated disorder behaves qualitatively different from short-range correlated disorder (α)†. We thus expect this also to be true in our case of a bounded support of $\delta\tau$. In the following we do not consider white noise disorder any further.

(II) *The distribution of the primary stochastic variables near its lower edge.* Let us specify the following three cases using a model of type A.

(i) The primary stochastic variables q_i take their lowest value ($q_i = 0$) with *finite probability* ($\neq 0$): $\mathrm{Prob}(q_i = 0) > 0$. The probability density ρ_q of q_i thus contains a delta function $\delta(q)$. This case is realized, for instance, in models with a discrete distribution of the q_i.

(ii) The distribution of primary stochastic variables exhibits *power law behaviour*: $\mathrm{Prob}(q_i \leq \epsilon) \sim \epsilon^\gamma$ for $\epsilon \to 0$, $\gamma > 0$. That means that the probability density ρ_q of the q_i behaves like $\rho_q(q) \sim q^{\gamma-1}$ for $q \to 0$.

(iii) The distribution of the primary stochastic variables exhibits an *exponential behaviour near the lower edge*: $\mathrm{Prob}(q_i \leq \epsilon) \sim \exp\{-B\epsilon^{-\beta}\}$ for $\epsilon \to 0$, $B, \beta > 0$. In terms of the probability density we thus have: $\rho_q(q) \sim \exp\{-Bq^{-\beta}\}$ for $q \to 0$, neglecting an algebraic prefactor.

It is not difficult to formulate conditions in analogy to (i)–(iii) for models with an accountable set of primary stochastic variables (type B).

Now we are in a position to give some examples for the exponents ζ and s determining the asymptotic behaviour of the Lifshitz tail (4.2).

(α, i) *Short-range correlations and finite probability* of the primary stochastic variables to take their lowest value:

$$\zeta = d/2 \qquad s = 0. \tag{4.4}$$

For type A models this can be proved rigorously [24]. The condition of short-range (α) is fulfilled if the shape function $f(x)$ (see (4.3)) decays like $|x|^{-\alpha}, \alpha \geq d+2$. There are related models with a countable set of primary stochastic variables such as the so-called Poisson model [21] or lattice models [20, 25] showing behaviour (4.2) with the same exponents (4.4). The treatment of type B models is mathematically more involved. Here the condition of short-range correlations has to be expressed in terms of a 'φ-mixing condition' of the stochastic field $\delta\tau(x)$ — for details see [24, 26]. However, the corresponding results are compatible with (4.4) [24].

(α, ii) *Short-range correlations* and *power law behaviour* of the primary stochastic variables:

$$\zeta = d/2 \qquad s = 1. \tag{4.5}$$

Both type A models and similar models previously mentioned yield the asymptotic Lifshitz behaviour given by (4.2) and (4.5) [19, 21, 27]. For type B models, on the other hand, the known rigorous results are not strong enough to prove $s = 1$, but they are at least consistent with (4.5) [24].

† For Gaussian disorder the Lifshitz tails have asymptotically, for $E \to -\infty$, the form $\rho(E) \sim \exp\{-\text{constant} \times E^x\}$, where $x = 2 - d/2$ in the white noise case, while $x = 2$ for short-range correlations [21–23].

(α, iii) *Short-range correlations* and *exponential behaviour* of the primary stochastic variables:

$$\zeta = d/2 + \beta \qquad s = 0. \tag{4.6}$$

Concerning this exponential behaviour of the distribution of the primary stochastic variables only a paper treating a lattice model is known to the author [19]. But it seems plausible that the results are also valid for continuum models just as in cases (α, i) and (α, ii).

(β, ii) *Long-range correlations* and *power law behaviour* of the primary stochastic variables:

$$\zeta = \frac{d}{\alpha - d} \qquad s = 1. \tag{4.7}$$

A rigorous proof of (4.7) is available for type A models [24, 28].

Table 1. Exponents ζ and s of Lifshitz tails (4.2) depending on the range of correlations of disorder and the distribution of the primary stochastic variables. For further explanation see main text.

$s\backslash\zeta$	(i)	(ii)	(iii)
(α)	$0\backslash d/2$	$1\backslash d/2$	$0\backslash\frac{1}{2}d+\beta$
(β)	—	$1\backslash\frac{d}{\alpha-d}$	—

Table 1 summarizes the findings of this section. Cases (β, i) and (β, iii) may behave in analogy to (α, i) and (α, iii), respectively. However, so far there have been no results on such models.

5. An essential singularity

So far we have dealt with the integrated density of states $Z_0(\mu)$ for arbitrary $\tau \geq 0$ and given some examples suggesting the general form (4.2) in the limit $\mu \searrow \tau$. This implies the asymptotic form of the density of states itself.

$$\rho_0(\mu) \sim \begin{cases} \exp\{-\kappa|\ln(\mu-\tau)|^s(\mu-\tau)^{-\zeta}\} & \text{for } \mu \searrow \tau \\ 0 & \text{for } \mu \leq \tau \end{cases} \tag{5.1}$$

where we have again neglected the algebraic and logarithmic prefactors. It is this essential singularity in $\rho_0(\mu)$ which generates, via (3.3), the (essential) Griffiths singularity in the free energy density $\mathcal{F}_0(\tau)$ when T approaches the Griffiths temperature T_G i.e. $\tau \searrow 0$.

To see this we split from $\mathcal{F}_0$ the singular part and write

$$\mathcal{F}_0(\tau) = \mathcal{F}_0^{\text{reg}}(\tau) + \frac{1}{2}n \int_\tau^{\mu_0} \mathrm{d}\mu \, \ln\mu \; \phi(\mu-\tau) \exp\{-\kappa|\ln(\mu-\tau)|^s(\mu-\tau)^{-\zeta}\} \tag{5.2}$$

where $\mu_0 > \tau$ is an arbitrary cutoff and ϕ takes into account the possible algebraic or/and logarithmic prefactors neglected so far. In other words: $\phi(x)$ may be a sum of terms of the form $A_\nu x^{b_\nu}(\log x)^{c_\nu}$ with appropriate real numbers A_ν, b_ν and c_ν. In what follows we want to consider a typical contribution to the Griffiths singularity and, thus, confine ourselves to the treatment of one such term, i.e. we set $\phi(x) = Ax^b(\log x)^c$. After a shift of the integration variable $\mu \to \mu - \tau$ and neglecting the thereby emerging τ-dependence of the upper boundary of integration (a regular contribution to $\mathcal{F}_0$) this part of the free energy density reads

$$\hat{\mathcal{F}}_0(\tau) = \frac{1}{2} nA \int_0^{\mu_0} \mathrm{d}\mu \, \ln(\mu+\tau)\mu^b(\ln\mu)^c \exp\{-k|\ln\mu|^s\mu^{-\zeta}\}. \tag{5.3}$$

Next we show that the radius of convergence R of a Taylor series of $\hat{\mathcal{F}}_0(\tau)$ in τ around $\tau = 0$ is zero. For a power series $\sum_\nu a_\nu x^\nu$ this radius is given by $R = \lim_{\nu\to\infty} |a_\nu|/|a_{\nu+1}|$ [29]. The coefficients a_ν of the desired Taylor expansion $\hat{\mathcal{F}}_0$ are

$$\begin{aligned} a_\nu &= \frac{1}{\nu!}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^\nu \hat{\mathcal{F}}_{0|\tau=0} \\ &= \frac{(-1)^{\nu-1}}{2\nu} nA \int_0^{\mu_0} \mathrm{d}\mu \, \mu^{b-\nu}(\ln\mu)^c \exp\{-\kappa|\ln\mu|^s\mu^{-\zeta}\}. \end{aligned} \tag{5.4}$$

Using a generalized Laplace method for integrals of the type $\int \mathrm{d}t\, g(t)\exp\{h(t,x)\}$ in the limit $x \to \infty$ [30] we evaluate $|a_\nu|$ in the limit $\nu \to \infty$ and obtain to leading order in ν:

$$|a_\nu| \sim n\frac{(2\pi)^{1/2}}{\kappa^{3/2}\zeta^2}(-\ln\bar{\mu}_\nu)^{c-3s/2}\,\bar{\mu}_\nu^{1+3\zeta/2}\exp\{-\kappa(-\ln\bar{\mu}_\nu)^s\bar{\mu}_\nu^{-\zeta}[1-s-\zeta(-\ln\bar{\mu}_\nu)]\}. \tag{5.5}$$

Here $\bar{\mu}_\nu$ is the ν-dependent point where the integrand of (5.4) takes its maximum value and is implicitly given by the equation

$$\nu = b - \frac{c}{-\ln\bar{\mu}_\nu} + \kappa(-\ln\bar{\mu}_\nu)^{s-1}\bar{\mu}_\nu^{-\zeta}[s+\zeta(-\ln\bar{\mu}_\nu)]. \tag{5.6}$$

Unfortunately this is a transcendental equation. Thus we cannot solve it for $\bar{\mu}_\nu$. But it is $|a_\nu/a_{\nu+1}|$ not $|a_\nu|$ itself in which we are interested. To get the limiting behaviour of $|a_\nu/a_{\nu+1}|$ for $\nu \to \infty$ we define the functions $N(\bar{\mu}_\nu) := \nu$ according to (5.6) and its inversion $M(\nu) := \bar{\mu}_\nu$, such that $N(M(x)) \equiv x$, and expand $\bar{\mu}_{\nu+1} = M(\nu+1)$ around ν. Expressing derivatives of M in terms of the function N we arrive at

$$\bar{\mu}_{\nu+1} = \bar{\mu}_\nu + [N'(\bar{\mu}_\nu]^{-1} - [N'(\bar{\mu}_\nu)]^{-3}N''(\bar{\mu}_\nu) + \ldots \tag{5.7}$$

with

$$[N'(\bar{\mu}_\nu)]^{-1} = -\zeta^{-1}\frac{\bar{\mu}_\nu}{\nu}\{1 + \mathrm{O}(\nu^{-1}(-\ln\bar{\mu}_\nu)^{s-1}\bar{\mu}_\nu^{-\zeta})\}$$

$$[N'(\bar{\mu}_\nu)]^{-3}N''(\bar{\mu}_\nu) = \mathrm{O}(\bar{\mu}_\nu/\nu^2).$$

Using these results one can show that

$$R = \lim_{\nu\to\infty} |a_\nu/a_{\nu+1}| = 0 \tag{5.8}$$

for arbitrary κ, $\zeta > 0, s \geq 0$, and $A, b, c \in \mathbb{R}$. On the other side (5.3) and (5.4) show that $\hat{\mathcal{F}}_0(\tau = 0)$ as well as all its derivatives with respect to τ exist at $\tau = 0$ and are finite. Thus we conclude that *the free energy density has an essential singularity at* $\tau = 0$.

Finally we give one particular example for which the coefficients a_ν can be given explicitly, namely

$$\begin{aligned}\hat{\mathcal{F}}_0 &= \frac{1}{2} nA \int_0^{\mu_0} \mathrm{d}\mu\, \ln(\mu + \tau)\mu^{-2} \exp\{-\kappa/\mu\} \\ &= -\tfrac{1}{2} nA\kappa^{-1} \exp\{\kappa/\tau\} \mathrm{Ei}(-\kappa/\tau) + \text{reg. terms}\end{aligned} \tag{5.9}$$

which corresponds to the case $b = -2$, $c = 0$, $s = 0$, $\zeta = -1$. Here $\mathrm{Ei}(x)$ is the exponential integral function [31] and 'reg. terms' means terms which are regular for $\tau \searrow 0$. The asymptotic expansion of $\mathrm{Ei}(x)$ for $x \to \infty$ is [31]

$$\mathrm{Ei}(x) = \mathrm{e}^{-x} \sum_{\nu=1}^{n} (-1)^\nu (\nu - 1)! x^{-\nu} + R_n(x) \tag{5.10}$$

where $|R_n(x)| < n!|x|^{n-1}$. Inserting this into (5.9) we get the desired asymptotic series $\hat{\mathcal{F}}_0 = \sum_\nu \tilde{a}_\nu \tau^\nu$ with

$$\tilde{a}_\nu = \tfrac{1}{2} nA(\nu - 1)!(-\kappa)^{-\nu-1} \tag{5.11}$$

in agreement with (5.5) and (5.6).

6. Summary and outlook

The purpose of this paper has been to obtain some insight into the connection between Griffiths singularities in disordered systems like ferromagnets and Lifshitz band tails of the corresponding disordered quantum systems. To this end we considered the standard Ginzburg–Landau model (2.1). Using the Ornstein–Zernike approximation with respect to thermal fluctuations we gave the analytical expression for the free energy (3.3) as a functional of the density of states of a related Hamiltonian operator (2.5). Due to disorder the density of states ends in a Lifshitz tail which eventually produces the Griffiths singularity of the free energy. We emphasize here that this type of consideration avoids any approximation or dodge, like the replica trick, regarding disorder. A short synopsis of different kinds of disorder helped us to determine the generic form of the Lifshitz band tails (see (4.2) and (5.1)) depending mainly on two exponents, ζ and s (see table 1). For general ζ and s we were able to show that this form of Lifshitz tails induces an essential singularity in the free energy, if the temperature approaches the Griffiths temperature ($T \searrow T_{\mathrm{G}}$).

Finally let us remark on two desirable extensions: (i) the temperature region $T_c < T < T_{\mathrm{G}}$; and (ii) the inclusion of higher thermal fluctuations.

(i) Concerning the first topic there are, at least, two points that have to be noted. On the one hand it is necessary to introduce a magnetic field H since the line $H = 0$, $T_c < T < T_{\mathrm{G}}$ is a line of singular points of the free energy [1, 12]. Consequently the Griffiths singularities emerge for vanishing H. On the other hand a stability problem arises in the naïve Ornstein–Zernike approximation. To see this consider the inverse propagator $\tilde{G}^{-1}_{0,L,\mathrm{bc}}$ for $\tau < 0$, i.e. the operator generalizing $G^{-1}_{0,L,\mathrm{bc}}$ (see (2.5) and (4.1)). $\tilde{G}^{-1}_{0,L,\mathrm{bc}}$ has to be positive definite, even in the limit $H \to 0$. However there are clearly realizations of disorder such that $\delta\tau(x) + \tau$ is negative for arbitrarily large regions in space. To stabilize the system one has to look for a new saddle-point solution of the full ϕ^4-theory (2.1). The result for a finite system of size L is a function $M_L(x)$ depending on the realization $\{\delta\tau(x)\}$. $M_L(x, \{\delta\tau\})$ is, in a sense, a local 'spontaneous magnetization' giving a typical (i.e. the most probable) contribution to the partition integral. It has to be carefully distinguished from the thermal expectation value of the magnetization, which vanishes in any finite system. This 'magnetization' $M_L(x, \{\delta\tau\})$ leads to a positive definite inverse propagator

$$\tilde{G}^{-1}_{0,L,\mathrm{bc}} = G^{-1}_{0,L,\mathrm{bc}} + \frac{u}{6} M_L^2(x). \tag{6.1}$$

In other words: below T_{G} there exist regions show the characteristics of a 'ferromagnetic phase'. This happens although the infinite system is not in the ferromagnetic phase and thus has vanishing total magnetization. How to handle this 'magnetization' $M_L(x, \{\delta\tau\})$ to obtain the 'Lifshitz tail' of (6.1) for $L \to \infty$ is still an open question.

(ii) Another aim would be to go beyond mean field theory. Let us turn first to the (integrated) density of states $Z(\mu)$ of the exact inverse propagator G^{-1} including all thermal fluctuations neglected so far. From the proofs of behaviour (4.2), especially the evaluation of $\zeta (= d/2$, for all 'short-range models' (α, i–iii)), we can extract two main ingredients [20, 21]:

(a) the probability of large deviations from the mean is proportional to $\exp\{-\text{constant } \times L^d\}$; and

(b) the scaling behaviour of the low-lying eigenvalues of $G^{-1}_{0,L,\mathrm{bc}}$ is given by $\mu \sim L^{-2}$.

Including thermal fluctuations does not affect (a) but leads to a modified scaling behaviour in (b) such that $\mu \sim L^{-\gamma/\nu}$. This can be seen by considering a heuristic argument given by Bray [4]. The lowest-lying eigenvalues of G^{-1} correspond to localized states located near large pure clusters, i.e. regions without (or with little) disorder, of typical size L. A corresponding ground-state eigenvalue μ of G^{-1} behaves as a function of L like the inverse susceptibility χ_L^{-1} of that cluster. Using the familiar finite size scaling [32] $\chi_L \sim L^{\gamma/\nu}$ we end up with the suggested scaling form $\mu(L)$. After substitution of (b) by this scaling form $\mu \sim L^{-\gamma/\nu}$ and exploiting the scaling law $\gamma = \nu(2 - \eta)$ we obtain the asymptotic behaviour of the integrated density of states for short-range correlated disorder:

$$Z(\mu) \sim \exp\{-\text{constant } \times (\mu - \tau)^{-d/(2-\eta)}\} \tag{6.2}$$

for $\mu \searrow \tau$†. Notice that in the mean field approximation $\eta = 0$, so that the old (Lifshitz) result is reproduced.

A second step had to be an instruction as to how to improve the mean field expression for the free energy (3.3). The idea of replacing the mean field density of states $\rho_0(\mu)$ in (3.3) by the exact one $\rho(\mu)$ corresponding to $Z(\mu)$ (6.2) does not yield an improved expression for $\mathcal{F}$, not even in the lowest order in the coupling constant u as can be easily seen perturbatively. However, it may be true that $\int \mathrm{d}\mu\, \rho(\mu) \ln \mu$ delivers the correct leading essentially singular behaviour of the thermodynamic quantities for $T \searrow T_{\mathrm{G}}$, leaving a more satisfying treatment of the thermal fluctuations as a task for further investigations.

† Here we have not worried about logarithmic terms such as $|\ln(\mu - \tau)|^s$.

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